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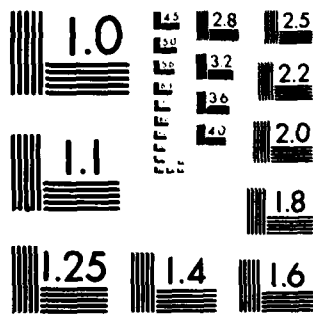
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THE ANSCOMBE-WOODROOFE METHOD IN RENEWAL THEORY

BY

STEVEN PAUL LALLEY

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# THE ANSCOMBE-WOODROOFE METHOD IN RENEWAL THEORY

Steven Paul Lalley, Ph.D.  
Stanford University, 1980

A probabilistic device developed by Anscombe and Woodroffe is used to obtain a local limit theorem for a class of hitting times associated with transient one-dimensional random walks. Applications of this local limit theorem to ruin problems and to nonlinear renewal theory are given.

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## THE ANSCOMBE-WOODROOFE METHOD IN RENEWAL THEORY

Many random walk problems (particularly those arising in sequential statistical analysis) call for precise information concerning both the hitting time and the hitting place of a half-line. If  $\{S_n\}$  is a random walk with positive drift  $\mu$  and finite variance  $\sigma^2$ , and if  $\tau_a = \min\{n : S_n > a\}$ , then it is well-known that  $S_{\tau_a} - a$  has a limiting distribution as  $a \rightarrow \infty$ , that  $(\tau_a - a\mu^{-1})\mu^{3/2}\sigma^{-1}\sigma^{-1/2}$  has approximately a standard normal distribution as  $a \rightarrow \infty$ , and that the two variables in question are asymptotically independent. Sometimes, however, stricter limiting statements are necessary: for example, a local limit theorem in  $\tau_a$ .

Such a theorem was obtained by A. Borovkov ([3]) under the assumption that the underlying distribution of the increments in the random walk be strongly nonlattice and have a finite two-sided Laplace transform. Borovkov's proof relies heavily on the complex-variable machinery related to the Wiener-Hopf method for solving integral equations. Unfortunately, the asymptotic representation for the hitting probabilities is not very explicit (it is not even apparent from Borovkov's theorem, for example, that  $(\tau_a - a\mu^{-1})a^{-1/2}$  has a limiting normal law).

A rather different approach to limit theorems for hitting times was discovered by F. J. Anscombe ([1]) in his study of sequential estimation procedures, and has recently been exploited by M. Woodroffe ([14], [15], [16]) in a variety of contexts, all of them



involving the crossing of curved boundaries. The essence of this method is a conditioned limit theorem for the random walk, which in turn is derived from a local limit theorem for the density of  $S_n$ . The conditioned limit theorem is then used to obtain unconditional renewal-type theorems, large deviation probabilities for various hitting times, or results concerning the moments of hitting times.

In the theorems to follow the method of Anscombe and Woodroffe will be adapted to local limit problems for hitting times. The usefulness of these local limit theorems will be illustrated in a very simple derivation of a large deviation theorem generalizing Cramér's ruin estimates. (To the best of my knowledge no such large deviation theorem has previously appeared in the literature.) In contrast to Woodroffe's work, no absolute continuity conditions will be imposed on the random walk: the only condition needed for the validity of the theorems is a finite second moment.

Local limit theorems for hitting times are also valid for a class of "perturbed" renewal processes. Such processes, introduced by Lai and Siegmund [ 7 ], are especially pertinent to sequential statistical analysis.

# 1. A Renewal Theorem Local in Time

Suppose that  $F$  is a nonlattice probability distribution with mean  $\mu > 0$  and variance  $\sigma^2 < \infty$ . (NOTE: An "arithmetic" distribution is a distribution whose support is contained in a discrete subgroup of  $\mathbb{R}$ ; a "lattice" distribution has support contained in a coset of a discrete subgroup.) Let  $X_1, X_2, \dots$  be iid from  $F$ , and  $S_n = X_1 + \dots + X_n$ .

STONE'S THEOREM (Stone [13]): As  $n \rightarrow \infty$

$$(1) \quad \epsilon_n \stackrel{\Delta}{=} \sup_{I: |I| \leq 1} \sup_{x \in R} \left( \begin{array}{l} \text{& ICK} \\ |n^{-1/2} P\{S_n - n\mu \in I + x\} - |I| \phi_\sigma(x/n^{1/2})| \end{array} \right) \rightarrow 0 .$$

Here the supremum is taken over all finite intervals ICK of length  $|I| \leq 1$  where  $K$  is a compact subset of  $R$ ; and  $\phi_\sigma(x) \stackrel{\Delta}{=} (2\pi\sigma^2)^{-1/2} e^{-x^2/2\sigma^2}$ .

Let  $\beta_n = \max_{m \geq n} \epsilon_m$ ,  $k$  a fixed integer,  $I$  a fixed finite interval, and  $\{a_n\}$  a sequence of real numbers. Define

$$(2) \quad F_{n,k,I}(dx_0, \dots, dx_k) = P\{X_n \in dx_0, \dots, X_{n-k} \in dx_k | S_n \in a_n + I\} .$$

THEOREM 1: If

$$(3) \quad \lim_{n \rightarrow \infty} \beta_{n-k} |I|^{-1} \exp\{(a_n - n\mu)^2/2n\sigma^2\} = 0 ,$$

then

$$(4) \quad F_{n,k,I} \xrightarrow{\Delta} F(dx_0) \dots F(dx_k) .$$

Furthermore, the convergence in (4) is uniform on any set

$\mathcal{J} = \{(\{a_n\}, I)\}$  of pairs such that  $\bigcup_{(\{a_n\}, I) \in \mathcal{J}} I$  is a bounded subset of  $\mathbb{R}$ , and such that the convergence in (3) is uniform over  $\mathcal{J}$ .

PROOF: Since

$$\begin{aligned} & F_{n,k,I}(dx_0, \dots, dx_k) \\ &= F(dx_0) \dots F(dx_k) P\{S_{n-k-1} \in a_n - \sum_{j=0}^k x_j + I\} / P\{S_n \in a_n + I\} , \end{aligned}$$

the theorem is an immediate consequence of Stone's Theorem. ///

This is the key to establishing the following renewal theorem.

Recall that

$$\tau_a \triangleq \min\{n : S_n > a\} .$$

THEOREM 2: Let

$$(8) \quad n_0 = n_0(u, a) = a\mu^{-1} + u\sigma\mu^{-3/2} a^{1/2} + o(a^{1/2}) ;$$

then

$$(9) \quad P\{\tau_a = n_0\} \sim e^{-u^2/2} \sigma^{-1} \mu^{3/2} (2\pi a)^{-1/2} ,$$

and

$$\begin{aligned} (10) \quad & P\{\tau_a = n_0; S_{\tau_a} - a \leq x\} \\ & \sim e^{-u^2/2} \sigma^{-1} \mu^{3/2} (2\pi a)^{-1/2} \int_0^x P\{S_{\tau_0} > y\} dy / ES_{\tau_0} \end{aligned}$$

as  $a \rightarrow \infty$ . These relations are valid uniformly for  $x$  bounded away from zero and  $u$  in any compact subset of  $\mathbb{R}$ , provided the  $o(a^{1/2})$  term in the definition of  $n_0$  is uniform

PROOF: Fix  $u \in \mathbb{R}$  and  $x \in (0, \infty)$ ; let  $r \in \mathbb{N}$  be a large but fixed number, and let  $I_j = ((j-1)x/r, jx/r]$ ,  $j = 1, \dots, r$ . Then

$$(11) \quad P\{\tau_a = n_0; S_{\tau_a} - a \leq x\} \\ = \sum_{j=1}^r P\{S_{n_0} \in a + I_j\} P\{\tau_a = n_0 | S_{n_0} \in a + I_j\}.$$

It is quite obvious that

$$(12) \quad 1 - P\{S_{n_0} - S_{n_0 - \ell} \leq jx/r, \text{ some } \ell, 1 \leq \ell \leq n_0 | S_{n_0} \in a + I_j\} \\ \leq P\{\tau_a = n_0 | S_{n_0} \in a + I_j\} \\ \leq 1 - P\{S_{n_0} - S_{n_0 - \ell} \leq (j-1)x/r, \text{ some } \ell, 1 \leq \ell \leq n_0 | S_{n_0} \in a + I_j\}.$$

In Lemma 2 to follow it will be shown that for each  $\epsilon > 0$  there is a  $k_1$  large enough that

$$(13) \quad \limsup_{a \rightarrow \infty} \max_{1 \leq j \leq r} P\{S_{n_0} - S_{n_0 - \ell} \leq x, \text{ some } \ell, n_0 - k_1 \leq \ell \leq n_0 | S_{n_0} \in a + I_j\} < \epsilon.$$

Moreover, by the Strong Law there is a  $k_2$  large enough that

$$(14) \quad P\{S_n \leq x, \text{ some } n \geq k_2\} < \epsilon.$$

Let  $k = \max(k_1, k_2)$ . Since Theorem 1 implies

$$(15) \quad P\{S_{n_0} - S_{n_0 - \ell} \leq \alpha, \text{ some } \ell, 1 \leq \ell \leq k | S_{n_0} \in a + I_j\}$$

$$\rightarrow P\{S_\ell \leq \alpha, \text{ some } \ell, 1 \leq \ell \leq k\}$$

for each  $\alpha \in (0, \infty)$ , it follows that

$$(16) \quad \sum_{j=1}^r P\{S_{n_0} \in a + I_j\} [1 - P\{S_\ell \leq jx/r, \text{ some } \ell\} - 2\epsilon]$$

$$\leq P\{\tau_a = n_0; S_\tau - a \leq x\}$$

$$\leq \sum_{j=1}^r P\{S_{n_0} \in a + I_j\} [1 - P\{S_\ell \leq (j-1)x/r, \text{ some } \ell\} + 2\epsilon] .$$

Relation (10) follows easily from this inequality. Stone's Theorem allows that

$$(17) \quad P\{S_{n_0} \in a + I_j\} \sim e^{-u^2/2} \sigma^{-1} \mu^{1/2} (2\pi a)^{-1/2} \cdot (x/r) ;$$

Lemma 1 to follow states that for all  $\alpha \geq 0$

$$1 - P\{S_\ell \leq \alpha, \text{ some } \ell \geq 1\} = P\{S_{\tau_0} > \alpha\} \mu / ES_{\tau_0} ;$$

and  $\epsilon > 0$  is arbitrary. Thus for large  $r$ , the two extreme sides of the inequality (16) both approximate the expression on the right hand side of (10).

Let  $K_1$  be a compact set of  $\mathbb{R}$  and  $K_2$  a compact set of  $(0, \infty)$ . To establish that the relation (10) is valid uniformly over  $u \in K_1$  and  $x \in K_2$ , it is enough to show that there is uniformity in (13), (14), (15), and (17). Uniformity in these relations follows immediately

from the statements of Lemma 2, the Strong Law, Theorem 1, and Stone's Theorem, respectively.

To complete the proof of the theorem it suffices to demonstrate that (9) holds uniformly for  $u$  in compact sets, for it will automatically follow from this and what we have already proved about (10) that (10) is valid uniformly for  $x$  bounded away from zero. In the proof of (9) we will appeal to the fact that (10) holds uniformly for  $x$  bounded away from zero and infinity.

Choose  $\epsilon > 0$  small, and let  $A > 0$  be large enough that

$$\sum_{j=0}^{\infty} [1 - F(A+j)] < \epsilon.$$

By Stone's Theorem there is a constant  $C > 0$  such that for each interval  $I$  of length 1 and all  $a$ ,

$$P\{S_{n_0-1} \in I\} \leq Ca^{-1/2}.$$

Now

$$\begin{aligned} & P\{\tau_a = n_0; S_\tau - a > A\} \\ & \leq \int_0^\infty P\{X_{n_0} > A+z\} P\{S_{n_0-1} \in a-dz\} \\ & \leq \sum_{j=0}^{\infty} [1 - F(A+j)] \cdot Ca^{-1/2} \\ & \leq \epsilon Ca^{-1/2}. \end{aligned}$$

We have already shown that

$$P\{\tau_a = n_0; S_{\tau_a} - a \leq A\} \sim e^{-u^2/2} \sigma^{-1} \mu^{3/2} (2\pi a)^{-1/2} \cdot \int_0^A P\{S_{\tau_0} > y\} dy / ES_{\tau_0} ;$$

since  $\epsilon > 0$  is arbitrary, (9) holds. This completes the proof. ///

LEMMA 1: For any random walk with mean  $\mu > 0$ , and for each  $\alpha > 0$ ,

$$(18) \quad 1 - P\{S_n \leq \alpha \text{ for some } n \geq 1\} = P\{S_{\tau_0} > \alpha\} \mu / ES_{\tau_0}$$

PROOF: This is an exercise in the use of the "Duality Principle" for random walks (cf. Feller [5], Chapter XII). Let

$$t \triangleq \inf\{n \geq 1 : S_n \leq 0\} .$$

Then

$$\begin{aligned} 1 - P\{S_n \leq \alpha, \text{ some } n \geq 1\} &= \int_{(\alpha, \infty)} P\{\min_{k \geq 0} S_k > \alpha - y\} F(dy) \\ &= \int_{(\alpha, \infty)} \sum_{k=0}^{\infty} P\{\min_{0 \leq j \leq k} S_j = S_k > \alpha - y\} P\{t = \infty\} F(dy) \\ &= \int_{(\alpha, \infty)} \sum_{k=0}^{\infty} P\{\tau_0 > k; S_k > \alpha - y\} F(dy) P\{t = \infty\} \\ &= P\{S_{\tau_0} > \alpha\} P\{t = \infty\} \\ &= P\{S_{\tau_0} > \alpha\} / E\tau_0 \\ &= P\{S_{\tau_0} > \alpha\} \mu / ES_{\tau_0} . \quad /// \end{aligned}$$

LEMMA 2: Let  $K_1$  be a compact set of  $\mathbb{R}$  and  $K_2$  a compact set of  $(0, \infty)$ .

Then under the conditions of Theorem 2,

$$(19) \inf_{k>0} \limsup_{a \rightarrow \infty} \sup_{u \in K_1} \sup_{x \in K_2} \max_{1 \leq j \leq r}$$

$$P\{S_{n_0} - S_{n_0 - \ell} \leq x, \text{ some } \ell, n_0 - k \leq \ell \leq n_0 | S_{n_0} \in a + I_j\} = 0 \quad .$$

PROOF: "Looking backwards" along sample paths one sees that

$$\begin{aligned} & P\{S_{n_0} - S_{n_0 - \ell} \leq x, \text{ some } \ell, n_0 - k \leq \ell \leq n_0 | S_{n_0} \in a + I_j\} \\ & \leq P\{S_\ell \leq x, \text{ some } \ell, k \leq \ell \leq n_0\} \\ & \cdot P\{S_{n_0} \in a + I_j | S_\ell < x, \text{ some } \ell, k \leq \ell \leq n_0\} / P\{S_{n_0} \in a + I_j\} \quad . \end{aligned}$$

By Stone's Theorem and the definitions of  $n_0$  and  $I_j$ ,

$$\sup_{u \in K_1} \sup_{x \in K_2} \max_{1 \leq j \leq r} [1/P\{S_{n_0} \in a + I_j\}] = O(a^{-1/2}) \quad .$$

By the Strong Law,

$$\inf_{k>0} \sup_{x \in K_2} P\{S_\ell < x, \text{ some } \ell \geq k\} = 0 \quad .$$

Thus to prove (19) it suffices to show that

$$(20) \sup_{u \in K_1} \sup_{x \in K_2} \sup_{y < x} \max_{0 \leq \ell \leq n_0} P\{S_{n_0} \in a + I_j | S_\ell = y\} = O(a^{-1/2}) \quad .$$

Relation (20) follows from Chebyshev's Inequality and Stone's Theorem, quite painlessly. For  $n_0 \geq \ell \geq a^{1/4} + 1/2 \log a$ ,



$$P\{S_{n_0} \in a + I_j | S_\ell = y\}$$

$$\leq P\{S_{n_0 - \ell} > a - y\}$$

$$\leq (n_0 - \ell)\sigma^2 / [-y + \ell\mu - u\sigma\mu^{-1/2}a^{1/2} + o(a^{1/2})]^2$$

$$= o(a^{-1/2})$$

uniformly for  $u \in K$ , (provided the  $o(a^{1/2})$  in the definition of  $n_0 = n_0(u, a)$  is uniform). On the other hand,  $\ell < a^{1/4 + 1/2} \log a$  implies  $n_0 - \ell > a/2\mu$  (for sufficiently large  $a$  and  $u \in K_1$ ); Stone's Theorem implies that

$$\sup_{I: |I| < B} P\{S_n \in I\} = O(n^{-1/2})$$

so

$$\sup_{u \in K_1} \sup_{x \in K_2} \sup_{y < x} \max_{0 \leq \ell \leq a^{1/2 + 1/4} \log a} P\{S_{n_0} \in a + I_j | S_\ell = y\} = O(a^{-1/2}) \quad . \quad ///$$

Theorem 2 is especially useful in studying large deviation probabilities for hitting times. As an illustration, we present an extension of Cramér's classical estimates for the one-sided gambler's ruin problem (cf. Feller [5]; also Siegmund [11]).

Let  $X_1, X_2, \dots$  be i.i.d. random variables from a distribution  $F$  with a finite Laplace transform in some interval: this distribution will be thought of as a member of an exponential family of probability distributions  $\{F_\theta : \theta \in J\}$ , i.e.,

$$(21) \quad F_{\theta}(dx) = e^{\theta x - \psi(\theta)} F(dx)$$

$$e^{\psi(\theta)} = \int_{\mathbb{R}} e^{\theta x} F(dx) ,$$

and  $J$  is the largest interval on which the Laplace transform  $e^{\psi(\theta)}$  is finite. Denote by  $\mu_{\theta}$  and  $\sigma_{\theta}^2$  the mean and variance of  $F_{\theta}$ , so

$$(22) \quad \mu_{\theta} = (d/d\theta)\psi(\theta)$$

$$\sigma_{\theta}^2 = (d^2/d\theta^2)\psi(\theta) .$$

Note that these simple relations imply that  $\mu_{\theta} \uparrow$ , and that  $\psi(\theta) \uparrow$  on  $\{\theta : \mu_{\theta} > 0\} \triangleq J_+$  but  $\psi(\theta) \downarrow$  on  $\{\theta : \mu_{\theta} < 0\} \triangleq J_-$ .

**THEOREM 3:** Suppose  $F$  is a nonlattice distribution, and let  $\tau_a = \min\{n : S_n > a\}$ . If  $\theta > 0$ ,  $\theta \in J$ , and  $\psi(\theta) > 0$ , then as  $a \rightarrow \infty$

$$(23) \quad P_0\{\tau_a \leq a/\mu_{\theta}\} \sim e^{-\theta a + \lceil a/\mu_{\theta} \rceil \psi(\theta)} (2\pi a)^{-1/2} \\ \cdot \sigma_{\theta}^{-1} \mu_{\theta}^{3/2} (1 - e^{-\psi(\theta)})^{-1} . \\ \cdot \int_0^{\infty} e^{-\theta x} P_{\theta}\{S_{\tau_0} > x\} dx / E_{\theta} S_{\tau_0} .$$

If  $\theta \in J_+$  is in the interior of  $J$ , and if  $\psi(\theta) < 0$ , then as  $a \rightarrow \infty$ ,

$$(24) \quad P_0\{a/\mu_{\theta} < \tau_a < \infty\} \sim e^{-\theta a + \lceil a/\mu_{\theta} + 1 \rceil \psi(\theta)} (2\pi a)^{-1/2} \\ \cdot \sigma_{\theta}^{-1} \mu_{\theta}^{3/2} (1 - e^{\psi(\theta)})^{-1} \\ \cdot \int_0^{\infty} e^{-\theta x} P_{\theta}\{S_{\tau_0} > x\} dx / E_{\theta} S_{\tau_0} .$$

Note that  $\theta > 0$ ,  $\psi(\theta) > 0$  implies  $\mu_\theta > 0$ , so the statement (23) makes sense.

PROOF OF THEOREM 3: For  $\theta > 0$ ,  $\theta \in J$ ,  $\psi(\theta) > 0$ ,

$$\begin{aligned} P_0\{\tau_a \leq a/\mu_\theta\} &= \int_{\{\tau_a \leq a/\mu_\theta\}} e^{-\theta S_\tau + \tau\psi(\theta)} dP_\theta \\ &= e^{-\theta a + \lfloor a/\mu_\theta \rfloor \psi(\theta)} \\ &\quad \cdot \sum_{j=0}^{\lfloor a/\mu_\theta \rfloor} \int_{\{\tau_a=j\}} e^{-\theta(S_\tau - a) + (j - \lfloor a/\mu_\theta \rfloor)\psi(\theta)} dP_\theta. \end{aligned}$$

For  $\lfloor a/\mu_\theta \rfloor - a^{1/3} \leq j \leq \lfloor a/\mu_\theta \rfloor$ , the approximation

$$P_\theta\{S_{\tau_a} - a \leq x | \tau_a = j\} \sim \int_0^x P_\theta\{S_{\tau_0} > y\} dy / E_\theta S_{\tau_0}$$

$$P_\theta\{\tau_a = j\} \sim (2\pi a)^{-1/2} \sigma_\theta^{-1} \mu_\theta^{3/2}$$

hold uniformly by Theorem 2, so

$$\begin{aligned} &\sum_{j=\lfloor a/\mu_\theta \rfloor - a^{1/3}}^{\lfloor a/\mu_\theta \rfloor} \int_{\{\tau_a=j\}} e^{-\theta(S_{\tau_a} - a) + (j - \lfloor a/\mu_\theta \rfloor)\psi(\theta)} dP_\theta \\ &\sim (2\pi a)^{-1/2} \sigma_\theta^{-1} \mu_\theta^{3/2} \int_0^\infty e^{-\theta x} P_\theta\{S_{\tau_0} > x\} dx / E_\theta S_{\tau_0} \\ &\quad \cdot \sum_{j=\lfloor a/\mu_\theta \rfloor - a^{1/3}}^{\lfloor a/\mu_\theta \rfloor} e^{(j - \lfloor a/\mu_\theta \rfloor)\psi(\theta)}. \end{aligned}$$

The last expression is a geometric series which converges to  $(1 - e^{-\psi(\theta)})^{-1}$  as  $a \rightarrow \infty$ . Since

$$\sum_{j=0}^{\infty} \left[ \frac{a}{\mu_{\theta}} \right]^{-a^{1/3}} \int_{\{\tau_a = j\}} e^{-\theta(S_{\tau_a} - a) + (j - \left[ \frac{a}{\mu_{\theta}} \right])\psi(\theta)} dP_{\theta} \leq e^{-a^{1/3}} \psi(\theta)$$

is of smaller order of magnitude than  $a^{-1/2}$ , the proof of (23) is complete.

The proof of (24) is quite similar, but a technical complication arises: if  $\theta < 0$  then  $e^{-\theta(S_{\tau} - a)}$  is an unbounded random variable. We will not reproduce the details of the argument here (however, see Example 1 of Section 2, where essentially the same problem of uniformity occurs); we have included Proposition 1 below to dispose of uniform integrability difficulties which arise in the use of Theorem 3. ///

PROPOSITION 1: Suppose  $g(y)$  is a positive nondecreasing function on  $[0, \infty)$ , that  $F$  is a distribution on  $\mathbb{R}$  with finite mean  $\mu > 0$ , and that

$$(25) \quad \int_0^{\infty} \int_{u \in [0, \infty)} g(u) F(du + z) dz < \infty.$$

Let  $\{T_{a,i} : i \in I, a \in \mathbb{R}\}$  be any family of stopping times for  $\{\sigma(X_1, \dots, X_n)\}_{n=1,2,\dots}$  such that if  $T_{a,i} = n$ , then  $S_{n-1} \leq a \leq S_n$ . Then

$$(26) \quad \sup_{i \in I} \sup_{a \in \mathbb{R}} E_F g(S_{T_{a,i}} - a) < \infty .$$

If  $F$  has a finite variance, then

$$(27) \quad \sup_n \sup_{a \in \mathbb{R}} n^{\frac{1}{2}} E_F [g(S_n - a); S_{n-1} \leq a \leq S_n] < \infty .$$

Other than the variables  $\tau_a = \min\{n: S_n > a\}$ , many families of random times are subsumed by this proposition, for example

$$T = \min\{n: S_n > (2an)^{\frac{1}{2}}\} , \quad a > 0 .$$

The condition (25) is not difficult to verify, in many instances of interest. If, for example,  $g(y) = e^{\theta y}$ ,  $\theta > 0$ , and

$$(28) \quad \int_0^\infty g(y) F(dy) = \int_0^\infty e^{\theta y} F(dy) < \infty ,$$

then

$$\begin{aligned} (29) \quad & \int_0^\infty \int_{u \in [0, \infty)} g(u) F(du + y) dy \\ &= \int_0^\infty \int_{u \in [0, \infty)} e^{\theta(u+y)} F(du + y) e^{-\theta y} dy \\ &\leq \int_0^\infty e^{-\theta y} dy \int_{u \in [0, \infty)} e^{\theta u} F(du) \\ &< \infty . \end{aligned}$$

PROOF OF PROPOSITION 1: Let

$$(30) \quad G(z) = \int_{u \in (0, \infty)} g(u) F(du + z)$$

for  $z \geq 0$ ; since  $g$  is nondecreasing,  $G \uparrow$ . Let

$$(31) \quad U(I) = \sum_{k=0}^{\infty} P\{S_k \in I\} ;$$

by the renewal theorem there is a constant  $C$  such that

$$(32) \quad U(I) \leq C < \infty$$

for every interval  $I$  of length 1. But

$$\begin{aligned}
 (33) \quad E_F g(S_{T_{a,1}} - a) &= \int_0^{\infty} \int_0^{\infty} g(u) F(du+z) \sum_{k=0}^{\infty} P\{T_{a,1} > k, S_k \in a-dz\} \\
 &\leq \int_0^{\infty} G(z) U(a-dz) \\
 &\leq \sum_{k=0}^{\infty} G(k+1) U(a-k-1, a-k] \\
 &\leq \sum_{k=0}^{\infty} G(k+1) \cdot C \\
 &< \infty
 \end{aligned}$$

by (25).

If  $F$  has a finite variance, then by either Stone's Theorem or the Local Limit Theorem for lattice distributions there is a constant  $C^* < \infty$  such that for every interval  $J$  of length 1,

$$\sup_n n^{\frac{1}{2}} P_F\{S_n \in J\} \leq C^* .$$

Hence

$$\begin{aligned}
(34) \quad & E_F[g(S_n - a); S_{n-1} \leq a < S_n] \\
& \leq \sum_{k=0}^{\infty} G(k) P_F\{k \leq a - S_{n-1} \leq k+1\} \\
& \leq n^{-1/2} C^* \sum_{k=0}^{\infty} G(k) \quad . \quad ///
\end{aligned}$$

In deriving the analogue of Theorem 2 for lattice random walks Stone's Theorem may be replaced by the more well-known local limit theorem for lattice distributions. Although the argument is the same, the result is quite different.

Let  $\{S_n\}$  be a random walk with increments  $\{X_j\}$  i.i.d. from  $F$ , which has mean  $\mu > 0$  and variance  $\sigma^2 < \infty$ . Suppose  $F$  is supported by the coset  $h + \mathbb{Z}$ , where  $0 \leq h < 1$  is either zero or irrational; and suppose there is no integer  $k > 1$  such that  $h + k\mathbb{Z}$  supports  $F$ . Let

$$\begin{aligned}
(35) \quad & \tau_a = \min\{n \geq 1 : S_n > a\} \\
& T = \min\{n \geq 1 : S_n > 0\} \quad .
\end{aligned}$$

THEOREM 2<sup>\*</sup>: If

$$(36) \quad n_0 = n_0(u, a) = a\mu^{-1} + u\sigma\mu^{-3/2}a^{1/2} + o(a^{1/2})$$

and if  $h = 0$ , then for  $x \in \{1, 2, \dots\}$ ,

$$\begin{aligned}
(37) \quad & P(\tau_a = n_0; S_{\tau_a} - a = x - (a - \lfloor a \rfloor)) \\
& \sim e^{-u^2/2} \sigma^{-1} \mu^{3/2} (2\pi a)^{-1/2} P\{S_T \geq x\} / ES_T
\end{aligned}$$

and

$$(38) \quad P\{\tau_a = n_0\} \sim e^{-u^2/2} \sigma^{-1} \mu^{3/2} (2\pi a)^{-1/2}$$

as  $a \rightarrow \infty$ . If  $h \neq 0$ , then for  $x \in \{1, 2, \dots\}$

$$(39) \quad P\{\tau_a = n_0; S_{\tau_a} - a = x - (a - n_0 h - \llbracket a - n_0 h \rrbracket)\}$$

$$\sim e^{-u^2/2} \sigma^{-1} \mu^{3/2} (2\pi a)^{-1/2} (ES_T)^{-1}$$

$$\cdot P\{S_T \geq x - (a - n_0 h - \llbracket a - n_0 h \rrbracket)\}$$

and

$$(40) \quad P\{\tau_a = n_0\} \sim e^{-u^2/2} \sigma^{-1} \mu^{3/2} (2\pi a)^{-1/2} (ES_T)^{-1}$$

$$\cdot \sum_{x=1}^{\infty} P\{S_T \geq x - (a - n_0 h - \llbracket a - n_0 h \rrbracket)\}$$

as  $a \rightarrow \infty$ . These relations ((37)-(40)) hold uniformly for  $u$  in any compact subset of  $\mathbb{R}$ , provided the  $o(a^{1/2})$  term in the definition of  $n_0$  is uniform.

The surprise is (40): for although  $(\tau_a - a\mu^{-1})\sigma^{-1} \mu^{3/2} a^{-1/2}$  is (asymptotically) normally distributed, the (asymptotic) density of  $\tau_a$  is not the discretized normal density. However, one may easily deduce the global limit theorem from (40) by appealing to Weyl's equidistribution theorem of number theory (see, for example, Chandrasekharan [4]).

Now suppose that in addition to having support contained in  $h + \mathbb{Z}$  (and in no coarser lattice)  $F$  has a Laplace transform which is



finite in some open interval  $J$  containing zero. As before, let

$$F_{\theta}(dx) = e^{\theta x - \psi(\theta)} F(dx)$$

$$e^{\psi(\theta)} = \int_{\mathbb{R}} e^{\theta x} F(dx)$$

$$\mu_{\theta} = \int_{\mathbb{R}} x F_{\theta}(dx)$$

$$\sigma_{\theta}^2 = \int_{\mathbb{R}} (x - \mu_{\theta})^2 F_{\theta}(dx)$$

$$J_+ = \{\theta \in J : \mu_{\theta} > 0\}.$$

**THEOREM 3\***: Suppose  $h = 0$ . If  $\theta > 0$ ,  $\theta \in J$ , and  $\psi(\theta) > 0$ , then as  $a \rightarrow \infty$  through  $Z$

$$(41) \quad P_0\{\tau_a \leq a\mu_{\theta}\} \sim e^{-\theta a + \lfloor a/\mu_{\theta} \rfloor \psi(\theta)} \cdot (2\pi a)^{-1/2} \\ \cdot \sigma_{\theta}^{-1} \mu_{\theta}^{3/2} (1 - e^{-\psi(\theta)})^{-1} \\ \cdot \sum_{x=1}^{\infty} e^{-\theta x} P_{\theta}\{S_T \geq x\} / E_{\theta} S_T.$$

If  $\theta \in J_+$  is interior to  $J$  and  $\psi(\theta) < 0$ , then as  $a \rightarrow \infty$ ,

$$(42) \quad P_0\{a/\mu_{\theta} < \tau_a < \infty\} \sim e^{-\theta a + \lfloor a/\mu_{\theta} + 1 \rfloor \psi(\theta)} \cdot (2\pi a)^{-1/2} \\ \cdot \sigma_{\theta}^{-1} \mu_{\theta}^{+3/2} (1 - e^{\psi(\theta)})^{-1} \\ \cdot \sum_{x=1}^{\infty} e^{-\theta x} P_{\theta}\{S_T \geq x\} / E_{\theta} S_T.$$

Suppose  $h \neq 0$ . If  $\theta > 0$ ,  $\theta \in J$ , and  $\psi(\theta) > 0$ , then as  $a \rightarrow \infty$  through  $\mathbb{R}$ ,

$$\begin{aligned}
 (43) \quad P_0\{\tau_a \leq a\mu_\theta\} &\sim e^{-\theta a + \lfloor a/\mu_\theta \rfloor \psi(\theta)} \cdot (2\pi a)^{-1/2} \\
 &\cdot (\sigma_\theta^{-1} \mu_\theta^{3/2} / E_\theta S_T) \\
 &\cdot \sum_{j=1}^{\lfloor a/\mu_\theta \rfloor} \sum_{x=1}^{\infty} P_\theta\{S_T \geq x - (a - jh - \lfloor a - jh \rfloor)\} \\
 &\cdot \exp\{-\theta[x - (a - jh - \lfloor a - jh \rfloor)]\} \\
 &+ (j - \lfloor a/\mu_\theta \rfloor) \psi(\theta) \}.
 \end{aligned}$$

If  $\theta \in J_+$  is interior to  $J$  and  $\psi(\theta) < 0$ , then as  $a \rightarrow \infty$  through  $\mathbb{R}$ ,

$$\begin{aligned}
 (44) \quad P_0\{a/\mu_\theta < \tau_a < \infty\} &\sim e^{-\theta a + \lfloor a/\mu_\theta + 1 \rfloor \psi(\theta)} (2\pi a)^{-1/2} \\
 &\cdot (\sigma_\theta^{-1} \mu_\theta^{3/2} / E_\theta S_T) \\
 &\cdot \sum_{j=\lfloor a/\mu_\theta + 1 \rfloor}^{\infty} \sum_{x=1}^{\infty} P_\theta\{S_T \geq x - (a - jh - \lfloor a - jh \rfloor)\} \\
 &\cdot \exp\{-\theta[x - (a - jh - \lfloor a - jh \rfloor)]\} \\
 &+ (j - \lfloor a/\mu_\theta \rfloor) \psi(\theta) \}.
 \end{aligned}$$

Although the series in (43) and (44) have a somewhat menacing aspect, they are "subgeometric": to obtain reasonable approximations one would need only a small number of terms.

## 2. Perturbed Renewal Processes

In recent years an extension of the renewal theory for random walks to a class of "perturbed" random walks has been effected (principally by M. Woodroffe ([14]) and T. L. Lai and D. Siegmund ([7], [8])); the germination of this new theory was triggered by the peculiar needs of sequential analysis. It is the object of this section to derive analogues of Theorem 2 appropriate for the more general setting of Lai and Siegmund's papers.

Let  $\{X_1\}$  be iid from a nonlattice distribution  $F$  with mean  $\mu > 0$ , variance  $\sigma^2$ , and finite third moment, and  $S_n = X_1 + \dots + X_n$  (the need for a third moment stems from the necessity of appealing to certain refinements of the Central Limit Theorem). Let

$$(45) \quad Z_n = S_n + \xi_n$$

$$T = T_a = \min\{n : Z_n > a\} \quad ;$$

here  $\{\xi_n\}$  is a sequence of random variables such that  $\xi_n$  is independent of the future  $\sigma(X_{n+1}, X_{n+2}, \dots)$ . Certain assumptions on the rate of growth and oscillation of the sequence  $\{\xi_n\}$  are necessary to obtain any results of a renewal-theoretic character: those that follow are of necessity more stringent than those of Lai and Siegmund, since more refined limit theorems are at stake.

ASSUMPTIONS ON  $\{\xi_n\}$ . There exist constants  $\eta \in (0, 1/4)$ ,  $\beta \in [0, 1)$ ,  $\gamma > 0$  such that for every  $\epsilon > 0$

$$A. \quad P\left\{ \max_{0 \leq n \leq r} |\xi_n| > r\mu\beta^{-1}(1 - \beta - \gamma) \right\} = o(r^{-1/4 + \eta/2})$$

$$B. \quad P\left\{ \max_{\beta r \leq n \leq r} |\xi_n| > r^{1/2 + \eta} \epsilon \right\} = o(r^{-1/4 + \eta/2})$$

$$C. \quad P\left\{ |\xi_r| > r^{1/4 + \eta/2} \epsilon \right\} = o(r^{-1/2})$$

$$D. \quad P\left\{ \max_{1 \leq k \leq r^{1/2 + \eta}} |\xi_r - \xi_{r+k}| > \epsilon \right\} = o(r^{-1/2})$$

Roughly speaking, (A) and (B) guarantee that  $T_a$  is not very much smaller than its expectation with any appreciable probability (see Lemma 3); (C) insures that the normal approximation to the distribution of  $Z_m$  is sufficiently sharp; and (D) implies that the process  $\{Z_n\}$  acts like a random walk for reasonable stretches of time. Not all interesting processes are ruled out by these assumptions, as the next result shows.

PROPOSITION 2. Suppose  $S_n = Y_1 + \dots + Y_n$  is a  $p$ -dimensional random walk whose increments  $\{Y_j\}$  are iid from a distribution with mean vector  $\mu$  and finite absolute  $2\ell$ -th moment, for some  $\ell \geq 2$ . Let  $\{a_n\}$  be a sequence of constants for which  $\max_{k \leq n} 3/4 |a_n - a_{n+k}| \rightarrow 0$ , and let  $g: \mathbb{R}^p \rightarrow \mathbb{R}$  be a function which is  $C^3$  in a neighborhood of  $\mu$  and satisfies

$$(46) \quad \max_{|x| \leq R} |g(x)| = O(R^\ell)$$

as  $R \rightarrow \infty$ . If

$$Z_n = n g(S_n/n) + a_n$$

and

$$\xi_n = Z_n - n g(\mu) - \langle S_n - n\mu | \nabla g(\mu) \rangle ,$$

then  $\{\xi_n\}$  satisfies Assumptions (A)-(D), for any  $\eta \in (0, 1/16)$  and  $\beta \in (0, 1)$ .

PROOF. The appropriate tool is S. Nagaev's generalization of Chebyshev's inequality (this is Corollary 2 of [10], a paper which contains a wealth of information concerning the accuracy of the normal approximation, including an improvement of the Berry-Esseen Theorem which will be used later in this section). Nagaev's Inequality states that if  $V_1, V_2, \dots$  are iid random variables with zero mean and if  $W_n = V_1 + \dots + V_n$ , then

$$(47) \quad P\{W_n > x\} < B_m E|V_1|^m n/x^m$$

for all  $x > 4\{n \max(0, \log(n^{m/2-1}/K_m E|V_1|^m))\}^{1/2}$  where

$$K_m = 1 + (m+1)^{(m+2)} e^{-m} ,$$

and  $B_m$  is an absolute constant depending only on  $m$ . Using (47) (with  $m=3$ ) in conjunction with Taylor's Theorem, one may easily show that  $\{\xi_n\}$  satisfies Assumption (C).

Assumption (D) is somewhat stickier. Nagaev's Inequality shows that for each  $\delta > 0$ ,

$$(48) \quad P\{ \max_{\beta r \leq n \leq r} |S_n - n\mu| > n\delta \} = o(r^{-1/2})$$

and on the complementary event

$$(49) \quad |\xi_n - \xi_{n+k}| \leq |(2n)^{-1} \langle S_n - n\mu | \nabla^2 g(\mu) | S_n - n\mu \rangle \\ - (2(n+k))^{-1} \langle S_{n+k} - (n+k)\mu | \nabla^2 g(\mu) | S_{n+k} - (n+k)\mu \rangle| \\ + \alpha n \{ |S_n/n - \mu|^3 + |S_{n+k}/(n+k) - \mu|^3 \} \\ + o(1)$$

for some constant  $\alpha > 0$  (this follows from Taylor's Theorem and the fact that the sequence  $\{a_n\}$  is slowly varying). The cubic term falls easily to (47) (with  $m=4$ ), and

$$(50) \quad |(2n)^{-1} \langle S_n - n\mu | \nabla^2 g(\mu) | S_n - n\mu \rangle \\ - (2(n+k))^{-1} \langle S_{n+k} - (n+k)\mu | \nabla^2 g(\mu) | S_{n+k} - (n+k)\mu \rangle| \\ \leq |[(2n)^{-1} - (2(n+k))^{-1}] \langle S_n - n\mu | \nabla^2 g(\mu) | S_n - n\mu \rangle| \\ + |\langle S_{n+k} - S_n - k\mu | \nabla^2 g(\mu) | S_n/n - \mu \rangle| \\ + |\langle S_{n+k} - S_n - k\mu | \nabla^2 g(\mu) | S_{n+k} - S_n - k\mu \rangle| / 2n .$$

The first and third terms may be easily disposed of by using (47) ( $m=4$ ); (47) also works on the second term, but only after a maximal inequality has been employed. The proper maximal inequality may be

stated as follows: if  $U_1, U_2, \dots$  are iid random variables with mean zero and finite variance, then there is a  $c > 0$  such that for all  $\epsilon > 0$ ,

$$P\{\max_{1 \leq k \leq n} (U_1 + \dots + U_k) > \epsilon\} \leq c P\{(U_1 + \dots + U_n) > \epsilon\}.$$

Assumptions (A) and (B) are relatively easy to verify, using the standard maximal inequality for  $L^2$  reverse martingales, and Taylor's Theorem. For (A) some caution is necessary since for  $n$  small,  $P\{|S_n - n\mu| > n\delta\}$  need not be small; this is the reason for the  $2k$ -th moment condition and the growth condition on  $g$ . Because the details of the verification are somewhat mundane, they are omitted.

The main result of this section is that the "local" renewal theorem generalizes to perturbed random walks.

**THEOREM 4.** Suppose the sequence  $\{\xi_n\}$  satisfies Assumptions 1 and 2, and that  $Z_n = S_n + \xi_n$ , where  $\{S_n\}$  is a random walk from a nonlattice distribution with mean  $\mu > 0$ , variance  $\sigma^2$ , and finite absolute third moment. Let

$$(51) \quad n_0 = n_0(u, a) = a\mu^{-1} + u\sigma\mu^{-3/2} a^{1/2} + o(a^{1/2}).$$

Then

$$(52) \quad P\{T_a = n_0\} \sim e^{-u^2/2} \sigma^{-1} \mu^{3/2} (2\pi a)^{-1/2}$$

and

$$(53) \quad P\{T_a = n_0; Z_T - a \leq x\}$$

$$\sim e^{-u^2/2} \sigma^{-1} \mu^{3/2} (2\pi a)^{-1/2} \int_0^x P\{S_{\tau_0} > y\} dy / ES_{\tau_0}$$

as  $a \rightarrow \infty$ . These relations are valid uniformly for  $u \in K_1$ ,  $x \in K_2$ , where  $K_1$  is a compact subset of  $\mathbb{R}$  and  $K_2 \subset (0, \infty)$ ,  $0 \notin \bar{K}_2$ , provided the  $o(a^{1/2})$  term in the definition of  $n_0$  is uniform.

One could attempt to prove Theorem 4 by following the pattern established in the proof of Theorem 2; however, many of the technical details can be sidestepped by incorporating the result of Theorem 2 into the proof. This is accomplished by first conditioning on

$$(54) \quad \mathfrak{F}_{n_1} \triangleq \sigma(X_1, \dots, X_{n_1}; \xi_1, \dots, \xi_{n_1}),$$

where

$$n_1 = n_1(a) = \lfloor a\mu^{-1} - a^{1/2} + \eta \rfloor.$$

Because of Assumptions (A) and (B),  $T_a > n_1(a)$  with preponderant probability (cf. Lemma 3 to follow); on the other hand, Assumption (D) guarantees that  $\max_{n_1 \leq n \leq n_0} |\xi_n - \xi_{n_1}|$  is small, so conditional on  $\mathfrak{F}_{n_1}$ ,  $T_a$  is nearly always the same as  $n_1 + \min\{n \geq 1 : S_n > a - Z_{n_1}\}$ , and  $Z_{T_a} - a$  is approximately  $S_{T_a} - (a - \xi_{n_1})$ . Thus Theorem 2 should allow for calculation of

$$(55) \quad P\{T_a = n_0; Z_T - a \leq x | \mathfrak{F}_{n_1}\}.$$

The remainder of the proof will consist of integrating out the conditional probability over the  $\sigma$ -algebra  $\mathfrak{F}_{n_1}$ , i.e., evaluating



$$E[P\{T_a = n_0; Z_T - a \leq x | \mathcal{F}_{n_1}\}] .$$

The conditional probability (55) is of appreciable size only when  $Z_{n_1}$  is in a certain interval whose length is roughly proportional to  $a^{1/4 + \eta/2}$  (which is in turn proportional to the standard deviation of  $S_{n_0} - S_{n_1}$ ). On this interval  $S_{n_1}$  has (approximately) a density which is (approximately) a constant multiple of Lebesgue measure (by Stone's Local Limit Theorem ; Assumption (C) insures that integrating out against  $S_{n_1}$  is not markedly different from integrating out against  $Z_{n_1}$ ).

The strategy of the proof is indicated schematically by Figure A1.

LEMMA 3. Suppose  $\{Z_n\}$ ,  $\{S_n\}$ , and  $\{\xi_n\}$  satisfy the conditions of Theorem 4. Let  $\epsilon > 0$ ,  $b > 0$  be fixed real numbers, and suppose  $J_a$  is any interval satisfying

$$(56) \quad |J_a| \leq ba^{1/4 + \eta/2}$$

and

$$(57) \quad (a - x_a) \geq \epsilon a^{1/2 + \eta}$$

where

$$x_a = \sup\{x : x \in J_a\} .$$

Then

$$(58) \quad P\{T_a \leq n_1; Z_{n_1} \in J_a\} = o(a^{-1/4 + \eta/2})$$

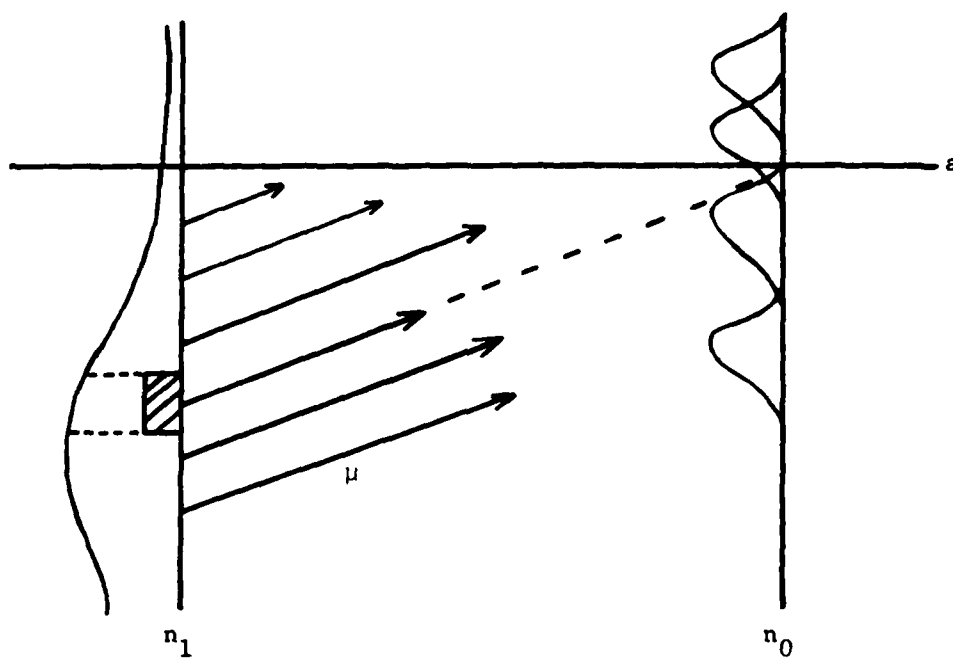


Figure A1

as  $a \rightarrow \infty$ . This holds uniformly for the family  $\{J_a\}$  of all intervals satisfying (56) and (57).

PROOF. This is essentially the same as the proof of Lemma 2: the perturbation terms  $\{\xi_n\}$  are the only complicating factor.

Let

$$I_{+\delta} \stackrel{\Delta}{=} \{y \in \mathbb{R} : \text{dist}(y, I) \leq \delta\}$$

for each interval  $I \subset \mathbb{R}$ , and let  $\Gamma_1, \Gamma_2, \Gamma_3$  be the events

$$\Gamma_1 = \{ \max_{0 \leq n \leq \beta n_1} |\xi_n| \leq a(1 - \beta - \gamma) \}$$

$$\Gamma_2 = \{ \max_{\beta n_1 \leq n \leq n_1} |\xi_n| \leq a^{1/2 + \eta} \epsilon/2 \}$$

$$\Gamma_3 = \{ |\xi_{n_1}| \leq a^{1/4 + \eta/2} \}.$$

Then by assumptions (A)-(C) on  $\{\xi_n\}$ ,

$$P(\Omega \setminus (\Gamma_1 \cap \Gamma_2 \cap \Gamma_3)) = o(a^{-1/4 + \eta/2}),$$

since  $n_1 = n_1(a) \sim a\mu^{-1}$ . Now

$$\{T_a \leq n_1; Z_{n_1} \in J_a; \Gamma_1 \cap \Gamma_2 \cap \Gamma_3\}$$

$$\subset \{ \min_{\beta n_1 \leq n \leq n_1} S_{n_1} - S_n < -\epsilon a^{1/2 + \eta/2}/2 + a^{1/4 + \eta/2} \};$$

$$S_{n_1} \in (J_a)_{+a^{1/4+\eta/2}}$$

$$\cup \{ \max_{0 \leq n \leq \beta_{n_1}} S_n > a(\beta + \gamma); S_{n_1} \in (J_a)_{+a^{1/4+\eta/2}} \} .$$

The probabilities of the two events on the RHS of this last relation may be bounded using the Strong Law, Chebyshev's Inequality, and Stone's Theorem, in much the same fashion as in the proof of Lemma 2. ///

Lemma 3 will justify conditioning on  $\mathfrak{J}_{n_1}$ , effectively allowing us to replace  $T_a$  by  $\min\{n > n_1 : Z_n > a\}$ . This was the first step in the strategy; the second was to integrate out over  $\mathfrak{J}_{n_1}$ . The next lemma provides a device for doing so. Recall that

$$n_0(u, a) = a\mu^{-1} + a^{1/2} \sigma \mu^{-3/2} \cdot u + o(a^{1/2})$$

$$n_1(a) = \llbracket a\mu^{-1} - a^{1/2+\eta} \rrbracket ;$$

define

$$(59) \quad k = k(u, a) = n_0(u, a) - n_1(a) = a^{1/2+\eta} - a^{1/2} \sigma \mu^{-3/2} u + o(a^{1/2}) .$$

LEMMA 4. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded uniformly continuous function such that  $\int_{-\infty}^{\infty} |f(x)| dx < \infty$ . Let

$$(60) \quad \omega(y) = [k - (a - y)\mu^{-1}] \mu k^{-1/2} ;$$

let  $A < B$  be any two real numbers (finite); and let  $\{\alpha_n\}$ ,  $\{\beta_n\}$  be sequences of constants converging to zero. Then if  $\{S_n\}$ ,  $\{\xi_n\}$ , and  $\{Z_n\}$  satisfy the hypotheses of Theorem 4,

$$\begin{aligned}
(61) \quad & \int_{\omega(y)=A+\alpha_{n_1}}^{B+\beta_{n_1}} f(\omega(y)) P\{Z_{n_1} \in dy\} \\
& \sim \int_{\omega=A}^B f(\omega) d\omega \cdot e^{-u^2/2} \mu^{1/2} k^{1/2} (2\pi a)^{-1/2} \sigma^{-1}
\end{aligned}$$

as  $a \rightarrow \infty$ . This holds uniformly for  $u$  in a bounded subset of  $\mathbb{R}$ , provided the  $o(a^{1/2})$  term in the definition of  $n_0(u, a)$  is uniform.

PROOF. This is a consequence of Stone's Local Limit Theorem for the random walk  $\{S_n\}$ , together with Assumption (C). Let  $\psi(\cdot)$  be the modulus of continuity of  $f$ , i.e.,

$$\psi(\delta) = \sup_{u, v: |u-v| \leq \delta} |f(u) - f(v)|,$$

and let  $\Gamma_\delta$  be the event

$$\Gamma_\delta = \{|\xi_{n_1}| \leq \delta k^{1/2}\}.$$

Now

$$\begin{aligned}
& \int_{\omega(y)=A+\alpha_n+\delta}^{B+\beta_n-\delta} [f(\omega) - \psi(\delta)] P\{S_{n_1} \in dy; \Gamma_\delta\} \\
& \leq \int_{\omega(y)=A+\alpha_n}^{B+\beta_n} f(\omega(y)) P\{Z_{n_1} \in dy; \Gamma_\delta\} \\
& \leq \int_{\omega(y)=A+\alpha_n-\delta}^{B+\beta_n+\delta} [f(\omega) + \psi(\delta)] P\{S_{n_1} \in dy; \Gamma_\delta\}.
\end{aligned}$$

By assumption (C),  $P\{\Omega \setminus \Gamma_\delta\} = o(a^{-1/4 + \eta/2})$ ; since  $f$  is bounded,

$$\begin{aligned}
& \int_{\omega(y)=A+\alpha_{n_1}-\delta}^{B+\beta_{n_1}+\delta} [f(\omega) - \psi(\delta)] P \{S_{n_1} \in dy\} + o(a^{-1/4 + \eta/2}) \\
& \leq \int_{\omega(y)=A+\alpha_{n_1}}^{B+\beta_{n_1}} f(\omega(y)) P \{Z_{n_1} \in dy\} \\
& \leq \int_{\omega(y)=A+\alpha_{n_1}-\delta}^{B+\beta_{n_1}+\delta} [f(\omega(y)) - \psi(\delta)] P \{S_{n_1} \in dy\} + o(a^{-1/4 + \eta/2}) .
\end{aligned}$$

The integrals on the extreme sides of the last inequality may be approximated using Stone's Theorem. Since  $\delta > 0$  was arbitrary, (61) follows: the uniformity in  $u$  follows directly from the statement of Stone's Theorem. ///

Recall that the strategy for evaluating  $P\{T_a = n_0(u, a); Z_T - a \leq x\}$  called for a careful analysis of

$$P\{T_a = n_0(u, a); Z_T - a \leq x; T_a > n_1(a); Z_{n_1} \in I(a, u)\}$$

where  $I(a, u)$  is some "critical interval" (see Figure 1). Implementation of this strategy requires that we bound  $P\{T = n_0; Z_T - a \leq x; Z_{n_1} \notin I(a, u)\}$ ; since there is no local limit theorem (or Berry Esseen bound) available (a priori) for the process  $\{Z_n\}$ , this is a somewhat delicate point.

**LEMMA 5.** Suppose the processes  $\{S_n\}$ ,  $\{\xi_n\}$ , and  $\{Z_n\}$  satisfy the hypotheses of Theorem 4. Then for bounded sets  $K \subset \mathbb{R}$ , and each  $B > 4 + b$ ,

$$(62) \quad \sup_{I: |I| \leq b} \sup_{u \in K}$$

$$\begin{aligned} & a^{1/2} P\{Z_{n_0} - n_0 \mu \in I; |Z_{n_0} - Z_{n_1} - (n_0 - n_1)\mu| > a^{1/4 + \eta/2} B\} \\ & \leq C(\mu, \sigma, m_3) \sum_{j=0}^{\infty} (2+b) [\phi_{\sigma}(B + 2j(b+1)) + (B - 4 + j(b+2))^{-3}] \\ & + a^{1/2} P\{|\xi_{n_1}| > a^{1/4 + \eta/2}\} \\ & + a^{1/2} P\{|\xi_{n_0} - \xi_{n_1}| > 1\} \end{aligned}$$

where

$$\phi_{\sigma}(x) = e^{-x^2/2\sigma^2} / \sqrt{2\pi} \sigma.$$

Here  $C(\mu, \sigma, m_3)$  is a constant depending only on the first three moments of  $S_1$ , and the supremum is over real intervals  $I$ . In addition there are constants  $C_{b,K}^*$  (depending on  $b, K$  and on the processes  $\{S_n\}, \{\xi_n\}$ ) such that

$$(63) \quad \sup_{a < \infty} \sup_{u \in K} \sup_{I: |I| \leq b} a^{1/2} P\{Z_{n_0} - n_0 \mu \in I\} \leq C_{b,K}^* ;$$

and for each  $B < \infty$  there is a constant  $C_{B,K}^{**}$  such that

$$(64) \lim_{a \rightarrow \infty} \sup_{u \in K} \sup_{I: |I| \leq \delta}$$

$$a^{1/2} P\{Z_{n_0} - n_0 \mu \in I; |Z_{n_0} - Z_{n_1} - (n_0 - n_1)\mu| \leq a^{1/4 + \eta/2} B;$$

$$|\xi_{n_1}| \leq \delta a^{1/4 + \eta/2}; |\xi_{n_0} - \xi_{n_1}| \leq \delta\}$$

$$\leq C_{B,K}^{**} \cdot \delta$$

for sufficiently small  $\delta > 0$ .

NOTE: The notation  $\sup_{u \in K}$  may be somewhat confusing, since  $u$  does not explicitly appear in any of the expressions. However, recall that  $n_0$  is defined as an integer such that

$$n_0(u, a) = a\mu^{-1} + a^{1/2} \sigma \mu^{-3/2} \cdot u + o(a^{1/2})$$

Implicit in the inequalities (59)-(61) is the assumption that the  $o(a^{1/2})$  term is uniform for  $u \in K$ .

The proof of the lemma will rely on a strengthening of the Berry-Esseen Theorem due to Nagaev [10].

NAGAEV'S NORMAL APPROXIMATION THEOREM. Let  $U_n = Y_1 + \dots + Y_n$  where  $Y_1, \dots$  are i.i.d. random variables with mean zero, variance 1, and finite absolute third moment  $m_3$ . Then

$$|P\{U_n \leq xn^{1/2}\} - \Phi(x)| \leq L m_3 n^{-1/2} (1 + |x|^3)^{-1}$$

for every  $n \in \mathbb{N}$  and  $x \in \mathbb{R}$ .  $L$  is a universal constant.



PROOF OF LEMMA 4. Let

$$I_{+\delta} = \{y \in \mathbb{R} : \text{dist}(y, I) \leq \delta\}$$

$$k = n_0 - n_1 = a^{1/2 + \eta} + u\sigma\mu^{-3/2} a^{1/2} + o(a^{1/2}).$$

Then

$$\begin{aligned} (65) \quad & P\{Z_{n_0} - n_0\mu \in I; |Z_{n_0} - Z_{n_1} - k\mu| > k^{1/2} B; |\xi_{n_1}| \leq k^{1/2}; |\xi_{n_0} - \xi_{n_1}| \leq 1\} \\ & \leq P\{S_{n_0} - n_0\mu + \xi_{n_1} \in I_{+1}; |S_{n_0} - S_{n_1} - k\mu| > k^{1/2} B - 1; |\xi_{n_1}| \leq k^{1/2}\} \\ & = \int P\{S_k - k\mu \in I_{+1} - y; |S_k - k\mu| > k^{1/2} B - 1\} \\ & \quad \cdot P\{S_{n_1} + \xi_{n_1} - n_1\mu \in dy; |\xi_{n_1}| \leq k^{1/2}\} \\ & \leq \sum_{j=0}^{\infty} P\{B - 1 + 2j(b+1) \leq |S_k - k\mu| k^{-1/2} \leq B - 1 + 2(j+1)(b+1)\} \\ & \quad \cdot P\{S_{n_1} - n_1\mu \in k^{1/2} J(j, I)\} \end{aligned}$$

where for each  $j$ ,  $J(j, I)$  is the union of two intervals whose lengths do not exceed  $4(b+1)$ . By the (usual) Berry-Esseen Theorem there is a constant  $C^*(\sigma, m_3)$  such that

$$P\{S_{n_1} - n_1\mu \in k^{1/2} J\} \leq (k^{1/2}/n_1^{1/2})(|J| + 1)C^*(\sigma, m_3)$$

for all intervals  $J$  and all  $a > 0$  (and uniformly for  $u \in K$ , since  $k = k(u, a) \uparrow \infty$  uniformly as  $a \uparrow \infty$ ). The probabilities

$$P\{B-1+2j(b+1) \leq |S_k - k\mu| k^{-1/2} \leq B-1+2(j+1)(b+1)\}$$

may now be estimated using Nagaev's Theorem to complete the proof of (62): again, the fact that  $u \in K$  (bounded) implies that  $k = k(u, a) \uparrow \infty$  uniformly for  $u \in K$ , so the bound provided by Nagaev's Theorem is uniform in  $u$ .

By Assumptions (C) and (D) on the process  $\{\xi_n\}$ ,

$$a^{1/2} P\{|\xi_{n_1}| > a^{1/4 + \eta/2}\} \rightarrow 0$$

$$a^{1/2} P\{|\xi_{n_0} - \xi_{n_1}| > 1\} \rightarrow 0.$$

Thus in view of (62) it suffices to show that  $P\{Z_{n_0} - n_0\mu \in I; |Z_{n_0} - Z_{n_1} - k\mu| \leq k^{1/2} B; \Gamma\} = O(a^{-1/2})$  in order to prove (63): here  $\Gamma = \{|\xi_{n_1}| \leq a^{1/4 + \eta/2}\} \cap \{|\xi_{n_0} - \xi_{n_1}| \leq 1\}$ . Now

$$\begin{aligned} (66) \quad & P\{Z_{n_0} - n_0\mu \in I; |Z_{n_0} - Z_{n_1} - k\mu| \leq k^{1/2} B; \Gamma\} \\ & \leq P\{S_{n_0} - n_0\mu + \xi_{n_1} \in I_{+1}; |S_{n_0} - S_{n_1} - k\mu| \leq k^{1/2} B + 1; \Gamma\} \\ & = \int P\{S_k - k\mu \in I_{+1} - y; |S_k - k\mu| \leq k^{1/2} B + 1\} \\ & \quad \cdot P\{S_{n_1} + \xi_{n_1} - n_1\mu \in dy; \Gamma\}. \end{aligned}$$

By the Berry-Esseen Theorem there is a constant  $C^*$  such that

$$(67) \quad P\{S_k - k\mu \in I_{+1} - y\} \leq C^*(|I| + 2)k^{-1/2}$$

$$P\{S_{n_1} + \xi_{n_1} - n_1\mu \in k^{1/2} J; \Gamma\} \leq C^*(|J| + 3)k^{1/2}/n_1^{1/2}$$

for all intervals  $I$  and  $J$  (on  $\Gamma$ ,  $|\xi_{n_1}/k^{1/2}| \leq 1$ ); furthermore,  $\{y : I_{+1} - y \cap [-k^{1/2} B - 1, k^{1/2} B + 1] \neq \emptyset\}$  is an interval of length no greater than  $2k^{1/2} B + 4 + |I|$ . Hence (63) follows from (66) and

(67). (Uniformity in  $u$  again follows from the fact that

$k = k(u, a) \uparrow \infty$  uniformly, provided  $u \in K$  bounded.)

The proof of (64) is quite similar. ///

PROOF OF THEOREM 4. Fix  $B > 0$  and  $\delta > 0$  ( $B$  should be thought of as being quite large, so that the RHS of (62) is small, and  $\delta$  should be thought of as small). Recall the notation

$$k = k(u, a) = n_0(u, a) - n_1(a) = a^{1/2 + \eta} + a^{1/2} \sigma \mu^{-3/2} u + o(a^{1/2}),$$

and let

$$\Gamma_\delta = \{|\xi_{n_1}| \leq \delta k^{1/2}\}$$

$$G_\delta = \left\{ \max_{n_1 \leq n \leq n_0} |\xi_n - \xi_{n_1}| \leq \delta \right\}.$$

Then

$$(68) \quad P\{T_a = n_0; Z_T - a \leq x\}$$

$$\leq P\{T_a = n_0; Z_T - a \leq x; |Z_{n_0} - Z_{n_1} - k\mu| \leq Bk^{1/2}; \Gamma_\delta; G_\delta\}$$

$$\begin{aligned}
& + P\{Z_{n_0} \in (a, a+x]; |Z_{n_0} - Z_{n_1} - k\mu| > Bk^{1/2}\} \\
& + P\{\Gamma_\delta\} \\
& + P\{G_\delta\} .
\end{aligned}$$

According to assumptions (C) and (D), the last two probabilities are  $o(a^{-1/2})$ ; and  $P\{Z_{n_0} \in (a, a+x]; |Z_{n_0} - Z_{n_1} - k\mu| > Bk^{1/2}\}$  will be small in comparison to the first probability if  $B$  is large (cf. (62)).

Thus it is the first probability that is of primary interest.

Begin with the inequality

$$\begin{aligned}
(69) \quad & \int_{y=a-k\mu-Bk^{1/2}+x}^{a-k\mu+Bk^{1/2}-x} P\{T_a = n_0; \delta < Z_T - a \leq x; G_\delta | Z_{n_1} = y; T > n_1; \Gamma_\delta\} \\
& \cdot P\{Z_{n_1} \in dy; T > n_1; \Gamma_\delta\} \\
& \leq P\{T_a = n_0; Z_T - a \leq x; |Z_{n_0} - Z_{n_1} - k\mu| \leq Bk^{1/2}; \Gamma_\delta; G_\delta\} \\
& \leq \int_{y=a-k\mu-Bk^{1/2}-x}^{a-k\mu+Bk^{1/2}+x} P\{T_a = n_0; \delta < Z_T - a \leq x; G_\delta | Z_{n_1} = y; T > n_1; \Gamma_\delta\} \\
& \cdot P\{Z_{n_1} \in dy; T > n_1; \Gamma_\delta\} \\
& + P\{a < Z_{n_0} \leq a + \delta; |Z_{n_0} - Z_{n_1} - k\mu| \leq Bk^{1/2}; \Gamma_\delta; G_\delta\} .
\end{aligned}$$

It follows from (64) that if  $\delta > 0$  is small, then  $P\{a < Z_{n_0} \leq a + \delta; |Z_{n_0} - Z_{n_1} - k\mu| \leq Bk^{1/2}; \Gamma_\delta; G_\delta\}$  is small compared to the integrals on the extreme sides of the inequality (69) (recall that  $k^{1/2} \sim a^{1/4 + \eta/2}$ ).

and that  $u$  is restricted to the bounded set  $K$ ). Thus the bulk of the argument consists of approximating the integrals in (69).

Theorem 2 is the key to evaluating the conditional probabilities: the reduction to the framework of an unperturbed process is provided by the obvious inequalities

$$\begin{aligned}
 (70) \quad & P\{\tau_{a-y-\delta} = n_0 - n_1 = k; S_T - a + y + \delta \in (2\delta, x]\} \cdot 1\{\Gamma_\delta; T_a > n_1\} \\
 & \leq P\{T_a = n_0; \delta < Z_T - a \leq x; G_\delta | Z_{n_1} = y; T_a > n_1; \Gamma_\delta\} \cdot 1\{\Gamma_\delta; T_a > n_1\} \\
 & \leq P\{\tau_{a-y} = n_0 - n_1 = k; S_T - a + y \leq x + \delta\} \cdot 1\{\Gamma_\delta; T_a > n_1\}
 \end{aligned}$$

where  $\tau_b \triangleq \min\{j \geq 1 : S_j > b\}$ . Since  $k \sim a^{1/2 + \eta} \uparrow \infty$  Theorem 2 provides asymptotic approximations to the probabilities on the extreme sides of (70); moreover, since  $a - k\mu - Bk^{1/2} - x \leq y \leq a - k\mu + Bk^{1/2} + x$ , these approximations are uniform on the range of integration. Thus for large  $a$ ,

$$\begin{aligned}
 (71) \quad & e^{-\omega(y)^2/2\sigma^2} \sigma^{-1} \mu(2\pi k)^{-1/2} \left[ \int_{2\delta}^x P\{S_{\tau_0} > s\} ds / ES_{\tau_0} - \delta \right] \\
 & \cdot 1\{\Gamma_\delta; T_a > n_1\} \\
 & \leq P\{T_a = n_0; \delta < Z_T - a \leq x; G_\delta | Z_{n_1} = y; T_a > n_1; \Gamma_\delta\} \\
 & \cdot 1\{\Gamma_\delta; T_a > n_1\} \\
 & \leq e^{-\omega(y)^2/2\sigma^2} \sigma^{-1} \mu(2\pi k)^{-1/2} \left[ \int_0^{x+\delta} P\{S_{\tau_0} > s\} ds / ES_{\tau_0} + \delta \right] \\
 & \cdot 1\{\Gamma_\delta; T_a > n_1\}
 \end{aligned}$$

where

$$(72) \quad \omega(y) = [k - (a - y)\mu^{-1}] \mu k^{-1/2},$$

for all  $\omega = \omega(y)$  such that  $\omega \in [-B - xk^{-1/2}, B + xk^{-1/2}]$ .

Combining (69) and (71) gives

$$(73) \quad \int_{\omega(y) = -B + xk^{-1/2}}^{B - xk^{-1/2}} e^{-\omega(y)^2/2\sigma^2} P\{Z_{n_1} \in dy; \Gamma_\delta; T_a > n_1\} \\ \cdot (2\pi k)^{-1/2} \sigma^{-1} \mu \cdot \left[ \int_{2\delta}^x P\{S_{\tau_0} > s\} ds / ES_{\tau_0} - \delta \right] \\ \leq P\{T_a = n_0; Z_T - a \leq x; |Z_{n_0} - Z_{n_1} - k\mu| \leq Bk^{1/2}; G_\delta; \Gamma_\delta\} \\ \leq \int_{\omega(y) = -B - xk^{-1/2}}^{B + xk^{-1/2}} e^{-\omega(y)^2/2\sigma^2} P\{Z_{n_1} \in dy; \Gamma_\delta; T_a > n_1\} \\ \cdot (2\pi k)^{-1/2} \sigma^{-1} \mu \cdot \left[ \int_0^{x+\delta} P\{S_{\tau_0} > s\} ds / ES_{\tau_0} + \delta \right] \\ + P\{a < Z_{n_0} \leq a + \delta; |Z_{n_0} - Z_{n_1} - k\mu| \leq Bk^{1/2}; \Gamma_\delta; G_\delta\}.$$

Recall that  $P\{\Gamma_\delta\} = o(a^{-1/2})$ , and that

$$P\{T_a > n_1; Z_{n_1} - a + k\mu \in [-Bk^{1/2} - x, Bk^{1/2} + x]\} \\ = o(a^{-1/4 + \eta/2}) \\ = o((k/a)^{1/2})$$

(by Lemma 3; note that for  $u \in K$  bounded,  $k = k(u, a) \sim a^{1/2 + \eta}$  uniformly, so (57) holds). Consequently, the measures

$$P\{Z_{n_1} \in dy; \Gamma_\delta; T_a > n_1\}$$

may be replaced by  $P\{Z_{n_1} \in dy\}$ . Lemma 4 now applies; thus for large  $a$

$$\begin{aligned} (74) \quad & \left[ \int_{-B}^B e^{-\omega^2/2\sigma^2} d\omega / (2\pi)^{1/2} \sigma \right] \cdot \mu^{3/2} (2\pi a)^{-1/2} e^{-u^2/2} \sigma^{-1} \\ & \cdot \left[ \int_{2\delta}^x P\{S_{\tau_0} > s\} ds / ES_{\tau_0} - 2\delta \right] \\ & \leq P\{T_a = n_0; Z_T - a \leq x; |Z_{n_0} - Z_{n_1} - k\mu| \leq Bk^{1/2}; G_\delta; \Gamma_\delta\} \\ & \leq \left[ \int_{-B}^B e^{-\omega^2/2\sigma^2} d\omega / (2\pi)^{1/2} \sigma \right] \cdot \mu^{3/2} (2\pi a)^{-1/2} e^{-u^2/2} \sigma^{-1} \\ & \cdot \left[ \int_0^{x+\delta} P\{S_{\tau_0} > s\} ds / ES_{\tau_0} + 2\delta \right] \\ & + P\{a \leq Z_{n_0} \leq a + \delta; |Z_{n_0} - Z_{n_1} - k\mu| \leq Bk^{1/2}; \Gamma_\delta; G_\delta\} . \end{aligned}$$

Letting  $\delta \downarrow 0$  and then  $B \uparrow \infty$  proves (53).

A careful examination of the argument will reveal that (53) holds uniformly for  $u \in K$  and  $x \in K'$ , where  $K$  is a bounded subset of  $\mathbb{R}$  and  $K'$  is a compact subset of  $(0, \infty)$ . The proof of (52) is all that remains: it will be accomplished by the same device as the proof of (9) in Theorem 2. Choose  $\epsilon > 0$ ; let  $x$  be large enough that

$$\int_x^\infty P\{S_{\tau_0} > y\} dy / ES_{\tau_0} < \epsilon$$

$$\sum_{j=0}^{\infty} P\{X_1 > x + j - 1\} < \epsilon.$$

For sufficiently large  $a$ ,

$$P\{T_a = n_0; Z_T - a \leq x\} = e^{-u^2/2} \sigma^{-1} \mu^{3/2} (2\pi a)^{-1/2} \cdot \left[ \int_0^x P\{S_{\tau_0} > y\} dy / ES_{\tau_0} \pm \epsilon \right],$$

and by (63) of Lemma 5

$$\sup_{I: |I| \leq 2} a^{1/2} P\{Z_{n_0-1} - (n_0-1)\mu \in I\} \leq C_{2,K}^*$$

provided  $u \in K$ , a bounded subset of  $\mathbb{R}$ . By assumption (D),

$$a^{1/2} P\{|\xi_{n_0} - \xi_{n_0-1}| > 1\} < \epsilon.$$

Since

$$\begin{aligned} & P\{T_a = n_0; Z_T - a > x\} \\ & \leq \sum_{j=0}^{\infty} P\{X_1 > x + j + 1\} P\{a - 2j \geq Z_{n_0} \geq a - 2j - 2\} + P\{|\xi_{n_0} - \xi_{n_0-1}|\} \end{aligned}$$

it follows that

$$P\{T_a = n_0\} = e^{-u^2/2} \sigma^{-1} \mu^{3/2} (2\pi a)^{-1/2} [1 \pm 2\epsilon] \pm \epsilon (C_{2,K}^* + 1) a^{-1/2}$$

uniformly for  $u \in K$ . Since  $\epsilon > 0$  was arbitrary, (52) follows. This completes the proof of the Theorem. ///



EXAMPLE. Suppose that  $Y_1, Y_2, \dots$  are i.i.d.  $N(\mu, 1)$ ,  $U_n = \sum_{j=1}^n Y_j$ , and

$$(75) \quad T_a = \min\{n : U_n^2 > 2an\}.$$

Let  $t > 0$  be fixed; if  $\mu > (2/t)^{1/2}$ , then

$$(76) \quad P_\mu\{T > at\} \sim (2\pi a)^{-1/2} e^{-a[\mu t^{1/2} - \sqrt{2}]^2} (2t)^{-3/2}$$

$$\cdot \int_0^\infty e^{(\mu - (2/t)^{1/2})x} P_{(2t)^{-1/2}}\{U_{\tau_0} > x\} dx / E_{(2t)^{-1/2}} U_{\tau_0}$$

$$\cdot \left[ 1 - e^{(\mu(2t)^{-1/2} - \mu^2/2)} \right]^{-1}$$

$$\cdot \exp\{(\lfloor at + 1 \rfloor - at) \cdot (\mu(2t)^{-1/2} - \mu^2/2)\}$$

as  $a \rightarrow \infty$ .

The probability in (76) has some importance in statistics in that it is the Type II error probability for a "repeated  $\chi^2$ -test."

The approximation given above was obtained earlier by Siegmund [12]

in a somewhat different form. Verifying that the two forms are

equivalent is a routine exercise in the use of Lemma 1.

PROOF OF (88). By the fundamental identity of sequential analysis,

$$(77) \quad P_\mu\{T > at\} = \int_{\{T > at\}} e^{(\mu - (2/t)^{1/2})U_T + T(t^{-1} - \mu^2/2)} dP_{(2/t)^{1/2}} \\ = \int_{\{T > at\}} \exp\{(\mu - (2/t)^{1/2})(U_T - (2aT)^{1/2})\} \\ \cdot \exp\{T((1/t) - (\mu^2/2)) + (2aT)^{1/2}(\mu - (2/t)^{1/2})\} dP_{(2/t)^{1/2}}.$$

Now  $\{T > at\} = \{at < T \leq at + a^{1/3}\} \cup \{at + a^{1/3} < T\}$ . We will proceed by evaluating the integral on the event  $\{at < T \leq at + a^{1/3}\}$ , and then showing that the integral on  $\{at + a^{1/3} < T\}$  is of smaller order of magnitude.

On  $\{at < T \leq at + a^{1/3}\}$ ,

$$|(2aT)^{1/2} - a(2t)^{1/2} - (T - at)(2t)^{-1/2}| \leq C a^{-1/3}$$

for some constant  $C$  not depending on  $a$ . Thus

$$\begin{aligned} (78) \quad & \int_{\{at < T \leq at + a^{1/3}\}} \exp\{(\mu - (2/t)^{1/2})(U_T - (2aT)^{1/2})\} \\ & \exp\{T((1/t) - (\mu^2/2)) + (2aT)^{1/2}(\mu - (2/t)^{1/2})\} \\ & \cdot dP_{(2/t)^{1/2}} \\ & \sim \sum_{n=\lfloor at+1 \rfloor}^{\lfloor at+a^{1/3} \rfloor} \int_{\{T=n\}} \exp\{(\mu - (2/t)^{1/2})(U_T - (2aT)^{1/2})\} \\ & \exp\{(T - at)(\mu(2t)^{-1/2} - \mu^2/2)\} \\ & \exp\{-a[\mu t^{1/2} - \sqrt{2}]^2/2\} dP_{(2/t)^{1/2}}. \end{aligned}$$

Next we deduce from Theorem 5 that for  $at < n \leq at + a^{1/3}$ ,

$$\begin{aligned} (79) \quad & P_{(2/t)^{1/2}} \{T = n; U_T - (2aT)^{1/2} \leq x\} \\ & \sim (2\pi a)^{-1/2} t^{-2} (2t)^{-1/2} \int_0^x P_{(2t)^{-1/2}} \{U_{\tau_0} > y\} \\ & \cdot dy/E_{(2t)^{-1/2}} U_{\tau_0}. \end{aligned}$$

The relevant nonlinear process in this problem is

$$\begin{aligned} Z_n &= U_n^2/2n \\ &= (n/t) + (U_n - n(2/t)^{1/2})(2/t)^{1/2} \\ &\quad + (U_n - n(2/t)^{1/2})^2/2n \end{aligned}$$

with

$$\begin{aligned} S_n &= (2/t)^{1/2}(U_n - n(2/t)^{-1/2}) \\ \xi_n &= (U_n - n(2/t)^{1/2})^2/2n . \end{aligned}$$

That the conditions of Theorem 5 are satisfied follows from

Proposition 2, so

$$\begin{aligned} (80) \quad P_{(2/t)^{1/2}} \{T=n; Z_T - a \leq x\} \\ \sim (2\pi a)^{-1/2} t^{-2} (2t)^{-1/2} \int_0^x P_{(2/t)^{1/2}} \{S_{\tau_0} > y\} dy / E_{(2/t)^{1/2}} S_{\tau_0} \\ = (2\pi a)^{-1/2} t^{-2} (2t)^{-1/2} \int_0^x P_{(2t)^{-1/2}} \{(2/t)^{1/2} U_{\tau_0} > y\} \\ \cdot dy / E_{(2t)^{-1/2}} (2/t)^{1/2} U_{\tau_0} . \end{aligned}$$

But

$$\begin{aligned} Z_T - a &= U_T^2/2T - a \\ &= (U_T - (2aT)^{1/2})(U_T/2T + (a/2T)^{1/2}) ; \end{aligned}$$

for  $at < T \leq at + a^{1/3}$ ,  $|(a/2T)^{1/2} - (2t)^{-1/2}| < (2a^{2/3} t^2)^{-1}$ , and furthermore for any  $\epsilon > 0$ ,  $p > 0$ ,  $P_{(2/t)^{1/2}}\{|U_T/2T - (2t)^{-1/2}| > \epsilon\} = o(a^{-p})$  (by any one of a host of large deviation theorems). Therefore (79) follows from (80).

Proposition 1 implies that the family of random variables

$$\mathfrak{F}_\theta = \left\{ a^{1/2} e^{\theta(U_T - (2aT)^{1/2})} 1\{T_a = n > at\} \right\}$$

is uniformly integrable with respect to any of the measures  $P_\eta$ ,  $\eta > 0$ . Consequently, we may sum the geometric series in (78) to obtain

$$\begin{aligned} (81) \quad & \int_{\{at < T < at+a^{1/3}\}} \exp\{(\mu - (2/t)^{1/2})(U_T - (2aT)^{1/2})\} \\ & \exp\{T((1/t) - (\mu^2/2)) + (2aT)^{1/2}(\mu - (2/t)^{1/2})\} \\ & \cdot dP_{(2/t)^{1/2}} \\ & \sim \exp\{-a[\mu t^{1/2} - \sqrt{2}]^2/2\} (2\pi a)^{-1/2} t^{-2} (2t)^{-1/2} \\ & \int_0^\infty e^{(\mu - (2/t)^{1/2})x} P_{(2t)^{-1/2}}\{U_{\tau_0} > x\} \\ & \cdot dx/E_{(2t)^{-1/2}} U_{\tau_0} (1 - e^{(\mu(2t)^{-1/2} - \mu^2/2)})^{-1} \\ & \cdot \exp\{(\lfloor at+1 \rfloor - at)(\mu(2t)^{-1/2} - \mu^2/2)\} . \end{aligned}$$

To see that  $\int_{\{at+a^{1/3} \leq T\}}$  is of smaller order of magnitude, note that for  $n \geq at + a^{1/3}$  the sequence

$n((1/t) - (\mu^2/2)) + (2an)^{1/2}(\mu - (2/t)^{1/2})$  is decreasing in  $n$ . Thus

$$\int_{\{at+a^{1/3} \leq T\}} \leq \exp\{-(a+a^{1/3}/t)[\mu t^{1/2} - \sqrt{2}]^2/2\} \\ \cdot E_{(2/t)^{1/2}} \exp\{(\mu - (2/t)^{1/2})(U_T - (2aT)^{1/2})\}.$$

Proposition 1 guarantees that  $E_{(2/t)^{1/2}} \exp\{(\mu - (2/t)^{1/2})(U_T - (2aT)^{1/2})\}$  is bounded as  $a \rightarrow \infty$ . This completes the proof of (76). ///

There is a useful generalization of Theorem 4 for vector-valued random walks. Let  $Y_1, Y_2, \dots$  be i.i.d. random vectors in  $\mathbb{R}^d$  from a distribution with mean vector  $\mu$  and covariance matrix  $I$ , and let  $U_n = Y_1 + \dots + Y_n$ . Suppose  $\{\xi_n\}$  is a sequence of random variables such that for each  $n$ ,  $\xi_n$  is independent of the future  $\sigma(Y_{n+1}, Y_{n+2}, \dots)$ . Define

$$(82) \quad Z_n = \sigma U_n^{(1)} + \xi_n, \quad \sigma > 0$$

$$T = T_a = \min\{n \geq 1 : Z_n > a\}$$

(here  $y^{(1)}$  denotes the 1-th coordinate of the vector  $y$ ).

THEOREM 5. Suppose  $\mu^{(1)} = EY_1^{(1)} > 0$ ,  $Y_1^{(1)}$  has a (marginal) non-lattice distribution, and  $Y_1$  has a finite absolute fourth moment. If the sequence  $\{\xi_n\}$  satisfies assumptions (A)-(D), then

$$(83) \quad P\{T_a = n_0(u, a); Z_T - a \leq x; T^{-1/2}(U_T - T_\mu) \in A\}$$

$$\sim C(A, u) (\mu^{(1)})^{3/2} \sigma^{1/2} a^{-1/2} \int_0^x P\{S_{T_0} > s\} ds / ES_{T_0}$$

and

$$(84) \quad P\{T_a = n_0(u, a); T^{-1/2}(U_T - T_\mu) \in A\} \sim C(A, u) (\mu^{(1)})^{3/2} \sigma^{1/2} a^{-1/2},$$

where

$$(85) \quad C(A, u) = (2\pi)^{-d/2} \cdot \int_{\{y \in A; y^{(1)} = u\}} e^{-|y|^2/2} dy^{(2)} \dots dy^{(d)}$$

$$S_n = \sigma U_n^{(1)}$$

$$\tau_0 = \min\{n \geq 1 : S_n > 0\}$$

$$n_0 = n_0(u, a) = a\sigma^{-1}(\mu^{(1)})^{-1} - u\sigma^{-1/2}(\mu^{(1)})^{-3/2} a^{1/2} + o(a^{1/2}).$$

Here  $A$  is any bounded polyhedron in  $\mathbb{R}^d$  for which  $C(A, u) \neq 0$ .

Relations (83) and (84) hold uniformly for  $x$  bounded away from zero, and  $u$  in any bounded set  $K \subset \mathbb{R}$  such that

$$\inf_{u \in K} C(A, u) > 0.$$

Note that the hypotheses do not require the vector  $Y_1$  to have a nonlattice distribution in the usual sense: only the first coordinate  $Y_1^{(1)}$  need be nonlattice.

The proof of the theorem is quite similar to that of Theorem 4. The necessary modifications are really very minor. First, since  $Y_1$  has a finite absolute fourth moment, the sequence  $\{n^{-1/2}(U_n - n\mu)\}$  satisfies assumption (D) for every  $\eta \in (0, 1/4)$  (i.e.,

$$P\left\{\max_{1 \leq k \leq r^{1+\eta}} |n^{-1/2}(U_n - n\mu) - (n+k)^{-1/2}(U_{n+k} - (n+k)\mu)| > \epsilon\right\} = o(r^{-1/2})$$

for every  $\epsilon > 0$ ); consequently

$$P\{T_a = n_0; Z_T - a \leq x; T^{-1/2}(U_T - T_\mu) \in A\} \\ \sim P\{T_a = n_0; Z_T - a \leq x; n_1^{-1/2}(U_{n_1} - n_1\mu) \in A\}$$

where  $n_1 = \lfloor a\mu^{-1} - a^{1/2} + \eta \rfloor$  as before. The latter probability may be evaluated by conditioning on  $\mathcal{F}_{n_1}$  and proceeding as in the preceding theorem, with one change: namely, Lemma 4 must be altered to allow evaluation of

$$\int_{\omega(y)=A+\alpha_{n_1}}^{B+\beta_{n_1}} f(\omega(y)) P\{Z_{n_1} \in dy; n_1^{-1/2}(U_{n_1} - n_1\mu) \in A\} .$$

This is very easily accomplished by using the full force of Stone's multidimensional local limit theorem (Theorem 1 of [13]), and gives the result

$$\int_{\omega(y)=A+\alpha_{n_1}}^{B+\beta_{n_1}} f(\omega(y)) P\{Z_{n_1} \in dy; n_1^{-1/2}(U_{n_1} - n_1\mu) \in A\} \\ \sim C(A, u) (k/a)^{1/2} (\mu^{(1)}/\sigma)^{1/2} \int_{\omega=A}^B f(\omega) d\omega .$$

The rest of the argument is the same. ///

The lattice cases are not nearly so pleasant. Even global limit theorems for the joint distribution of  $T$  and  $Z_T - a$  reflect the dependence of the random walk  $\{S_n\}$  and the perturbation terms  $\xi_n$  (cf. [9]); local theorems will reflect it even more clearly.

Let  $Y_1, Y_2, \dots$  be i.i.d. random vectors in  $\mathbb{R}^d$  from a distribution with mean vector  $\mu$  and covariance matrix  $\Sigma$ , and let  $U_n = Y_1 + \dots + Y_n$ . Let  $Q: \mathbb{R}^d \rightarrow \mathbb{R}$  be a quadratic form, i.e.,

$$(86) \quad Q(x) = x^T M x$$

for some symmetric matrix  $M$ ; let  $\{\xi_n\}$  be a sequence of random variables satisfying Assumptions (A)-(D) such that each  $\xi_n$  is independent of the future  $\sigma(Y_{n+1}, \dots)$ , and such that for each  $\epsilon > 0$

$$(87) \quad P\{|\xi_n| > \epsilon\} = o(n^{-1/2}) ;$$

and let  $\{a_n\}$  be a sequence of constants satisfying

$$\max_{1 \leq k \leq n^{-3/4}} |a_n - a_{n+k}| \rightarrow 0. \text{ Define}$$

$$(88) \quad Z_n = U_n^{(1)} + Q(U_n - n\mu)n^{-1/2} + \xi_n + a_n$$

$$T = T_a = \min\{n \geq 1 : Z_n > a\}$$

and let  $r$  be a constant,  $0 \leq r < 1$ . Denote by  $\sigma^2$  the variance of  $Y_1^{(1)}$  (thus  $\sigma^2 = \Sigma_{11}$ ).

**THEOREM 6.** Suppose  $\mu^{(1)} = EY_1^{(1)} > 0$ ,  $Y_1^{(1)}$  has an arithmetic distribution of span 1 (i.e.,  $P\{Y_1^{(1)} \in \mathbb{Z}\} = 1$ ), and  $Y_1$  has a finite absolute fourth moment. Let  $f: [0, \infty) \rightarrow \mathbb{R}$  be a bounded continuous function. Then as  $a \rightarrow \infty$  through  $r + \mathbb{Z}$ ,



$$(89) \quad a^{1/2} E f(Z_{T_a} - a) 1\{T_a = n_0; (U_T - T\mu) T^{-1/2} \in A\}$$

$$= (\mu^{(1)})^{3/2} (2\pi)^{-d/2} (\det \frac{1}{4})^{-1/2} (EU_{\tau_0}^{(1)})^{-1}$$

$$\cdot \int_{\{y: y^{(1)} = u\sigma; y \in A\}} \left[ \sum_{j=0}^{\infty} f(j + [(Q(y) + a_{n_0} - r) \bmod 1]) \right. \\ \left. \cdot P\{U_{\tau_0}^{(1)} \geq j+1\} \right]$$

$$\cdot e^{-y^T \frac{1}{4}^{-1} y/2} dy^{(2)} \dots dy^{(d)} \rightarrow 0$$

for

$$(90) \quad n_0 = n_0(u, a) = a(\mu^{(1)})^{-1} - u\sigma^{-1}(\mu^{(1)})^{3/2} + o(a^{1/2})$$

$$\tau_0 = \min\{n \geq 1 : U_n^{(1)} > 0\}$$

and  $A$  any bounded polyhedron in  $\mathbb{R}^d$ . If instead of being arithmetic  $Y_1^{(1)}$  is supported by the lattice  $h + \mathbb{Z}$  for some irrational  $h \in (0, 1)$ , then as  $a \rightarrow \infty$  through  $r + \mathbb{Z}$ ,

$$(91) \quad a^{1/2} \operatorname{Ef}(Z_{T_a} - a) 1\{T_a = n_0; (U_T - T_\mu) T^{-1/2} \in A\}$$

$$= (\mu^{(1)})^{3/2} (2\pi)^{-d/2} (\det \frac{1}{4})^{-1/2} (EU_{\tau_0}^{(1)})^{-1}$$

$$\begin{aligned} & \cdot \int_{\{y: y^{(1)} = u\sigma; y \in A\}} \left[ \sum_{j=0}^{\infty} f(j + [Q(y) + a_{n_0} + n_0 h - r] \bmod 1) \right. \\ & \quad \cdot P\{U_{\tau_0}^{(1)} \geq j + [Q(y) + a_{n_0} + n_0 h - r] \bmod 1\} \left. \right] \\ & \cdot e^{-y^T \frac{1}{4} y} y^{T/2} dy^{(2)} \dots dy^{(d)} \rightarrow 0 \end{aligned}$$

The relations (89) and (91) are valid uniformly for  $u$  in any bounded subset of  $\mathbb{R}$ .

The proof of Theorem 6 will be omitted.

### 3. Remarks

Professor Siegmund has pointed out that the various local theorems derived in sections 1 and 2 may be used to obtain large deviations theorems of the Bahadur-Ranga Rao type (cf. [2]) for stopped sums, provided the increments have finite two-sided Laplace transform. For example, precise asymptotic expressions for

$$P\{S_{\tau_a} - \tau_a(EX_1) > \epsilon \tau_a\}$$

and

$$P\{S_{T_a} - T_a(EX_1) > \epsilon T_a\} \quad ,$$

where

$$\tau_a = \min\{n : S_n = X_1 + \dots + X_n > a\}$$

and

$$T_a = \min\{n : S_n > \sqrt{2a(c+n)}\}$$

are available for  $\{X_i\}$  i.i.d.  $n(\mu, 1)$ , exponential, uniform  $(0, 1)$ , etc., for all  $\epsilon > 0$ . The derivations are so simple that they are left for the amusement of the reader.

The primary impetus for this work was a desire to obtain asymptotic approximations for the Type II error probabilities of repeated significance tests in multiparameter exponential families. These results (which use Theorems 5 and 6) will appear elsewhere.

# REFERENCES

- [1] Anscombe, F. (1952). Large sample theory of sequential estimation. Proc. Cambridge Phil. Soc. 48, 600-607.
- [2] Bahadur, R. and Ranga Rao, R. (1960). On deviations of the sample mean. Ann. Math. Stat. 31, 1015-1027.
- [3] Borovkov, A. (1962). New limit theorems in boundary problems for sums of independent terms. Selected Translations in Math. Stat. and Probability 5, 315-372.
- [4] Chandrasekharan, K. (1968). Introduction to Analytic Number Theory. pp. 84-90. Springer-Verlag, New York.
- [5] Feller, W. (1966). An Introduction to Probability Theory and its Applications, Vol. II. Wiley, New York.
- [6] Hagwood, C. (1980). (unpublished manuscript).
- [7] Lai, T.L. and Siegmund, D. (1977). A nonlinear renewal theory with applications to sequential analysis I. Ann. Stat. 5, 946-954.
- [8] Lai, T.L. and Siegmund, D. (1979). A nonlinear renewal theory with applications to sequential analysis II. Ann. Stat. 7, 60-76.
- [9] Lalley, S. (1980). Nonlinear renewal theory for lattice random walks. Stanford Technical Report.
- [10] Nagaev, S. (1965). Some limit theorems for large deviations. Theory of Probability and its Applications 10, 214ff.
- [11] Siegmund, D. (1975). The time until ruin in collective risk theory. Mitt. Verein. Schweiz. Versich.-Math. 75, 157-166.
- [12] Siegmund, D. (1977). Repeated significance tests for a normal mean. Biometrika 64, 177-189.
- [13] Stone, C. (1967). On local and ratio limit theorems. Proc. Fifth Berkeley Symposium, Vol. II, part II, 217-224.
- [14] Woodroffe, M. (1976). A renewal theorem for curved boundaries and moments of first passage times. Ann. Prob. 4, 67-80.

- [15] Woodroffe, M. (1976). Frequentist properties of Bayesian sequential tests. Biometrika 63, 101-110.
- [16] Woodroffe, M. (1978). Large deviations of likelihood ratio statistics with applications to sequential testing. Ann. Stat. 6, 72-84.

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